FP2 hints - JAN 2011

Question 1

One of the standard loci is the circle, whose equation is $|z - z_1| = r$. Here, r is the radius of the circle, and is usually a real number, while z_1 is the centre of the circle, and is usually complex. You have to be a bit careful with signs in this question: you have to mould the equation given into the standard form, which means encasing the 4 and the 3i in appropriate brackets so as to be a single entity with a – in front of it.

I've used the symbol z_1 above, because that is what one always uses for the centre of the circle before sketching it. Don't confuse it with the z_1 of the question, which they have introduced after the circle has been sketched!

$\arg(z - z_1) = \theta$ is the equation of a straight line starting at z_1 and going on for ever in the direction of a line inclined at θ to the real (horizontal) axis. θ is usually expressed in radians. Again, don't confuse the z_1 of the standard equation with the particular z_1 of part (b). You'll need to identify the values of z_1 and θ before sketching this line on top of your sketch of the circle.

To answer (b)(ii) you must first identify the real and imaginary parts of the question's z_1 from your sketch. Then square them, add them and square root the sum.

Question 2

Write out the given expression for u_r, follow that with a – sign, and then write out the given expression for u again, but this time writing $(r - 1)$ wherever you find an r It is sometimes a good idea to include the brackets (as I have). It is never a bad idea! Then it's a question of algebra. I'd start by taking $^1/_6\, r$ out as a factor.

For part (b), look at the end of the given expression and compare it with the right-hand side of the equality you've just shown in (a). This will give you the clue to spot that the first term (1×5) can be written as $1(2\times1 + 3)$, which, as proved in (a) can be written as $u_1 - u_0$. So the whole expression can be written as

$$(u_1 - u_0) + (u_2 - u_1) + (u_3 - u_2) + \ldots\ldots\ (u_{99} - u_{98}) + (u_{100} - u_{99})$$

Now notice that most of the terms cancel out. Identify the two that remain, write them out in full using the definition given at the beginning of the question, and calculate.

Question 3

Expand $(1 + i)^3$ using the third row of Pascal's triangle as the coefficients, then replacing i^2 with -1 whenever it appears and finally collecting real and imaginary terms.

For part (b), you have to be a bit careful because it is NOT the case that one of the other roots is the coplext conjugate of (1 + i), that rule only applying to cubics with real coefficients.

The equation has to work for each of its three roots, so the thing to do is to plug in $z = (1 + i)$. Note that you've already evaluated the first term in part (a). Once again replace all i^2 terms with $-1i^2$ with -1 once you've done the preliminary algebra and then collect real and imaginary parts. Both parts will contain k and both parts have to be equal to zero by virtue of the = 0. This should give you k.

For part (ii) use the fact that $\alpha + \beta + \gamma = -b/a$, using the given value of α together with b and a derived from the coefficients of the equation.

For part (iii), use the fact given in (ii) to obtain β in terms of γ, and then substitute into $\alpha\beta\gamma = -d/a$ This will probably result in a quadratic in γ, which you can solve int he usual way.

Question 4

Start by expressing the equation in terms of e^x, using $\cosh x = ½(e^x + e^{-x})$ and $\sinh x = (e^x - e^{-x})$. Differentiate w.r.t. x, and set equal to 0 (the condition for a turning point). You now have an equation containing e^x and e^{-x}. The standard thing to do in this situation is to multiply all terms by e^x , which gets rid of the negative index, but leaves you with an e^{2x}. Express this as $(e^x)^2$, and then treat the equation as a quadratic in e^x: you could even substitute p for e^x throughout if you liked. Solve this quadratic in the usual way, and you will obtain two roots, one positive and one negative. But e to the something can't possibly be negative, so only one of the roots counts.

Substitute this value back into the original equation to find the values of a and b. You will need to use the fact that if $e^x = p$, then $x = \ln p$.

Question 5

(a) Write the x-expression as $(1 - x^2)^{½}$. Use the Chain Rule : i.e. differentiate the $()^{½}$, leaving the $1 - x^2$ intact inside it, and then multiply by the derivative of the $1 - x^2$.

(b) Observe from pp 7 & 8 of the tables that you can differentiate $\sin^{-1}x$, but you can't integrate it. So you will have to regard the integrand as $1\times \sin^{-1}x$, and aim to integrate the 1 while differentiating the $\sin^{-1}x$. I don't know how you think of Integration by Parts. I would integrate the 1 and multiply it by the $\sin^{-1}x$. Then I'd subtract the integral of (the integral of the $1\times$ the derivative of the $\sin^{-1}x$). This would leave me trying to integrate $x(1 - x^2)^{½}$, which I could do very easily using the substitution suggested in (a): you'll find that, once the substitution is made you are merely integrating $1\,\mathrm{d}u$, which is, of course, just u. Then I'd substitute back again before plugging in the limits.

Question 6

You begin by applying the Chain Rule, multiplying the derivative of ln() [i.e. 1/()] with the $\sec t + \tan t$ intact, and multiplying by the derivative of the $\sec t + \tan t$. You will find that this cancels down to just $\sec t$. Then you have to add in the derivative of $-\sin t$. Finding your way to the final answer involves replacing $\cos t$ with $\sec t \cos^2 t$, factorising, and then replacing 1 with $\sin^2 t + \cos^2 t$.

Part (b) is tables to the rescue! The third 'Arc Length' formula on page 8 is your starting point. Substitute in the expression for $\mathrm{d}x/\mathrm{d}t$ that you have just derived, and differentiate the given expression to obtain $\mathrm{d}y/\mathrm{d}t$. There's now some trig to do: you have to take $\sin^2 t$ out as a factor and then use $1 + \tan^2 t = \sec^2 t$, followed by $\sec^2 t = 1/\cos^2 t$. You should end up having to integrate $\tan t$, which you do by looking for the answer at the top of page 8 in the tables. Finally, use the 30/60/90 standard triangle for the limits.

Question 7

I hope that part (a) will just go by algebraic substitution. You'll write f(k + 1) as the top expression with (k + 1) instead of k, and you'll write 5f(k) as the top expression as it stands but multiplied by 5. The crucial step is to re-write 2×5^k as $2 \times 5 \times 5^{k-1}$, when the troublesome 5 to the something terms will obligingly disappear leaving only terms that are manifestly divisible by 12^k.

It's difficult to comment sensibly on part (b) without assuming the value of a from part (a), so I'l leave that to you.

Question 8

(a) Find the modulus of (1 + i √3), and take it out as a factor. That will leave (cosθ + i sinθ) from which you will easily determine θ by the use of the 30/60/90 triangle. So r is 4 × this modulus.

(b) Replace z^3 with p and then treat the equation as a quadratic to be expanded out and solved in the usual way. Or (quicker) take the square root of both sides of the equation (you know how to deal with the root of a negative number) and the answer appears in one move.

(c) So we are now trying to solve $z^3 = 4(1 + i\sqrt{3})$. Write $z = re^{i\theta}$. Cube this, remembering to cube the r as well as the $e^{i\theta}$. Set this equal to your answer to (a) and then argue that the modulus and argument parts of this equation are separately equal.

Once again, (d) depends on previous answers, and I can't sensibly discuss it.